DEFINITION AND EXAMPLES OF TOPOLOGIES

DEFINITION: Given a set X a collection of subsets $\mathcal{T} \subseteq \mathcal{P}(X)$ is called a **topology** on X if:

- $X, \emptyset \in \mathcal{T}$
- If $T_\alpha \in \mathcal{T}$ for $\alpha \in \Delta$, then $\bigcup_{\alpha \in \Delta} T_\alpha \in \mathcal{T}$

  NOTE: Another way of saying this is that $\mathcal{T}$ is 'closed under arbitrary unions.'
- If $T_1, T_2 \in \mathcal{T}$, then $T_1 \cap T_2 \in \mathcal{T}$.

  NOTE: Using induction, we can show this is equivalent to: if $T_k \in \mathcal{T}$ for $1 \leq k \leq n$ then $\bigcap_{k=1}^{n} T_k \in \mathcal{T}$.

  Another way of saying this is that $\mathcal{T}$ is 'closed under finite intersections.'

EXAMPLE: Let $X = \{1, 2, 3\}$ and $\mathcal{T} = \{\emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$ Show $\mathcal{T}$ is a topology on X.

EXAMPLE: Let $X = \{1, 2, 3\}$ and $\mathcal{T} = \{\emptyset, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$. Show $\mathcal{T}$ isn't a topology on X.

What set(s) must be added to $\mathcal{T}$ to make it a topology on X?

EXAMPLE: List all topologies for $X = \{1, 2, 3\}$.

HINT: There are 29.

QUESTION: Which of your topologies are 'sorta the same'?

EXAMPLE: For $n \in \mathbb{N}$, let $T_n = \{1, 2, \dots n\}$ and let $\mathcal{T} = \{T_n : n \in \mathbb{N}\}$.

- Show the T_n are **nested**. That is, prove that $T_n \subseteq T_m$ whenever $n \leq m$.

- Show that the intersection of any two elements of $\mathcal{T}$ is also in $\mathcal{T}$.

- Is it true that the arbitrary (non-empty) intersection of elements of $\mathcal{T}$ is also in $\mathcal{T}$?

  HINT: You may want to research the 'well ordering principle' of $\mathbb{N}$.

- Show that the union of any two elements of $\mathcal{T}$ is also in $\mathcal{T}$.

- Is it true that the arbitrary union of elements of $\mathcal{T}$ is also in $\mathcal{T}$?

- What subsets of $\mathbb{N}$ would need to be added to $\mathcal{T}$ so that $\mathcal{T}$ to get a topology on $\mathbb{N}$?

EXAMPLE: Let $X = \mathbb{N}$ and let $\mathcal{T}_{42} = \{T : T = \emptyset \text{ or } 42 \in T\}$. Show $\mathcal{T}_{42}$ is a topology on $\mathbb{N}$.

EXAMPLE: Let $X \neq \emptyset$ with $a \in X$. Let $\mathcal{T}_a = \{T : T = \emptyset \text{ or } a \in T\}$. Show $\mathcal{T}_a$ is a topology on X.

EXAMPLE: Let $X \neq \emptyset$ with $a \in X$. Let $\mathcal{U}_a = \{U : U = X \text{ or } a \notin U\}$. Is $\mathcal{U}_a$ is a topology on X?

DEFINITION: The elements of a topology are called **open** sets. A set C is **closed** iff $\tilde{C}$ is open.

EXAMPLE: Let $X = \mathbb{R}$. Define T to be open iff for all $x \in T$, there is an interval (a, b) so that $x \in (a, b) \subseteq T$.

Prove this defines a topology on $\mathbb{R}$. This topology is called the 'Euclidean' Topology on $\mathbb{R}$.

EXAMPLE: Let $X = \mathbb{R}$. Define T to be open iff for all $x \in T$, there is an interval $[a, b)$ so that $x \in [a, b) \subseteq T$.

Prove this defines a topology on $\mathbb{R}$. This topology is called the 'Sorgenfrey' Topology on $\mathbb{R}$.